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Exact solitary wave solutions of three-wave interaction equations with dispersion

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Abstract. The nonlinear evolution equations for three-wave resonances including intrapulse dispersion are solved by a special ansatz. Several types of three-wave solitary waves and kink solutions are provided explicitly.

Wave interactions and related solitary waves and solitons are fascinating phenomena widely occuring in fluids and solids, plasmas and nonlinear optics [1–3]. Resonant interaction is particularly interesting since it leads to an efficient energy exchange between waves with different frequencies. In a medium without a centre of symmetry, quadratic nonlinearity results in parametric three-wave mixing when two fundamental waves with the frequencies ω_1 and ω_2 create a harmonic wave with a combined frequency $\omega_3 = \omega_1 + \omega_2$. Dynamical equations describing such a mixing process are the well known three-wave interaction (TWI) equations without including intrapulse dispersion [4]. These equations are completely integrable and can be solved by the inverse scattering transform in a systematic way [5]. However, the TWI equations are not valid for fairly short pulses because in this case the effect due to the intrapulse dispersion plays a significant role. To include the intrapulse dispersion, the second-order derivatives with respect to time should be introduced into the usual TWI equations. In recent years there has been considerable interest shown in such processes but most of the studies concentrated only on the degenerate case $\omega_1 = \omega_2$, i.e. a second-harmonic generation [6].

In this paper, we consider the solitary wave solutions of the TWI equations with *dispersion* taken into account. We obtain some new solitary wave solutions and thus extend the work of Werner and Drummond [7].

We start with the TWI equations with dispersion in nonlinear optics [8]:

$$\mathrm{i}\left(\frac{\partial F_1}{\partial z} + \frac{1}{v_1}\frac{\partial F_1}{\partial t}\right) - g_1\frac{\partial^2 F_1}{\partial t^2} + \sigma F_2^* F_3 \mathrm{e}^{\mathrm{i}\Delta kz} = 0 \tag{1}$$

$$\mathrm{i}\left(\frac{\partial F_2}{\partial z} + \frac{1}{v_2}\frac{\partial F_2}{\partial t}\right) - g_2\frac{\partial^2 F_2}{\partial t^2} + \sigma \frac{\omega_2}{\omega_1} F_3 F_1^* \mathrm{e}^{\mathrm{i}\Delta kz} = 0 \tag{2}$$

$$\mathrm{i}\left(\frac{\partial F_3}{\partial z} + \frac{1}{v_3}\frac{\partial F_3}{\partial t}\right) - g_3\frac{\partial^2 F_3}{\partial t^2} + \sigma \frac{\omega_3}{\omega_1} F_1 F_2 \mathrm{e}^{-\mathrm{i}\Delta kz} = 0 \tag{3}$$

describing the interaction in a dispersive medium of three pulses with the following parameters: associated electric fields $E_j(t, z) = F_j(t, z)\exp[\mathrm{i}(\omega_j t - k_j z)] +$ c.c. $(j = 1, 2, 3)$; centre frequencies ω_j; group velocities v_j; second-order dispersion coefficients, i.e. group-velocity dispersion, g_j; wavenumbers $k_j = n_j\omega_j/c$, where c is the speed of light and n_j is the refractive index; nonlinear coupling constants $\sigma = 2\pi\chi_{\rm nl}\omega_1^2/(k_1c^2)$, where $\chi_{\rm nl}$ is the nonlinear dielectric susceptibility; and wavevector mismatch $\Delta k = k_3 - k_1 - k_2$. The usual TWI equations, i.e. when taking g_1, g_2 and g_3 to be zero in equations (1)–(3), are completely integrable. At variance with [8] when solving equations (1)–(3) we take the dispersion coefficients g_j to be nonzero.

Applying the transformation $s = T_0^{-1}(t - z/v_1)$, $\xi = z/L_D$, $F_j = F_{j0}a_j(s,\xi)$ with $F_{10} = \omega_1/(\sigma L_D\sqrt{\omega_2\omega_3})$, $F_{20} = \sqrt{\omega_1/\omega_3}/(\sigma L_D)$ and $F_{30} = \sqrt{\omega_1/\omega_2}/(\sigma L_D)$, where T_0 denotes the pulsewidth and L_D the dispersion length ($=T_0^2/|g_1|$), leads to the dimensionless form of the equations (1)–(3):

$$\mathrm{i}\frac{\partial a_1}{\partial\xi} - \frac{\alpha_1}{2}\frac{\partial^2 a_1}{\partial s^2} + a_2^* a_3 \mathrm{e}^{-\mathrm{i}\beta\xi} = 0 \tag{4}$$

$$\mathrm{i}\left(\frac{\partial a_2}{\partial\xi} - \gamma_2\frac{\partial a_2}{\partial s}\right) - \frac{\alpha_2}{2}\frac{\partial^2 a_2}{\partial s^2} + a_3 a_1^* \mathrm{e}^{-\mathrm{i}\beta\xi} = 0 \tag{5}$$

$$\mathrm{i}\left(\frac{\partial a_3}{\partial\xi} - \gamma_3\frac{\partial a_3}{\partial s}\right) - \frac{\alpha_3}{2}\frac{\partial^2 a_3}{\partial s^2} + a_1 a_2 \mathrm{e}^{\mathrm{i}\beta\xi} = 0 \tag{6}$$

where $\alpha_1 = \mathrm{sgn}(g_1)/2, \alpha_2 = g_2/|g_1|, \alpha_3 = g_3/|g_1|, \beta = -(\Delta k)L_D, \gamma_2 = -(L_D/T_0)(1/v_2 - 1/v_1)$ and $\gamma_3 = -(L_D/T_0)(1/v_3 - 1/v_1)$. γ_j $(j = 2, 3)$ are also called the temporal walk-off parameters. In order to obtain the vector solitary wave solutions of equations (4)–(6) we assume $a_j = U_j(\theta)\exp(\mathrm{i}\theta_j)$ with $\theta = \Omega s - K\xi$, $\theta_j = K_j\xi - \Omega_j s$ $(j = 1, 2, 3)$. Then equations (4)–(6) are transformed into the following ordinary differential equations:

$$-\frac{\alpha_1}{2}\Omega^2 U_1'' + U_2^* U_3 + \mathrm{i}(-K + \alpha_1\Omega\Omega_1)U_1' + \left(-K_1 + \frac{\alpha_1}{2}\Omega_1^2\right)U_1 = 0 \tag{7}$$

$$-\frac{\alpha_2}{2}\Omega^2 U_2'' + U_1^* U_3 + \mathrm{i}(-K + \alpha_2\Omega\Omega_2 - \gamma_2\Omega)U_2' + \left(-K_2 + \frac{\alpha_2}{2}\Omega_2^2 - \gamma_2\Omega_2\right)U_2 = 0 \tag{8}$$

$$-\frac{\alpha_3}{2}\Omega^2 U_3'' + U_1 U_2 + \mathrm{i}(-K + \alpha_3\Omega\Omega_3 - \gamma_3\Omega)U_3' + \left(-K_3 + \frac{\alpha_3}{2}\Omega_3^2 - \gamma_3\Omega_3\right)U_3 = 0 \tag{9}$$

with the conditions

$$K_3 = K_1 + K_2 + \beta \tag{10}$$

$$\Omega_3 = \Omega_1 + \Omega_2. \tag{11}$$

To solve equations (7)–(9) we use the ansatz [9]

$$U_j = A_j + B_j\,\mathrm{sech}\theta\tanh\theta + C_j\,\mathrm{sech}^2\theta \tag{12}$$

for $j = 1, 2, 3$, where A_j, B_j and C_j are generally complex constants yet to be determined. Substituting (12) into the equations (7)–(9) and equating the coefficients of the hyperbolic functions, we obtain

$$K/\Omega = \alpha_1\Omega_1 = \alpha_2\Omega_2 - \gamma_2 = \alpha_3\Omega_3 - \gamma_3 \tag{13}$$

as well as the nonlinear algebraic equations determing A_j, B_j and C_j, which are listed in the appendix.

Equations (11) and (13) yield

$$\Omega_1 = \frac{-\alpha_2\gamma_3 + \alpha_3\gamma_2}{\alpha_1\alpha_2 - \alpha_2\alpha_3 - \alpha_3\alpha_1} \tag{14}$$

$$\Omega_2 = \frac{\alpha_1(\gamma_2 - \gamma_3) - \alpha_3\gamma_2}{\alpha_1\alpha_2 - \alpha_2\alpha_3 - \alpha_3\alpha_1} \tag{15}$$

$$\Omega_3 = \frac{\alpha_1(\gamma_2 - \gamma_3) - \alpha_2\gamma_3}{\alpha_1\alpha_2 - \alpha_2\alpha_3 - \alpha_3\alpha_1}. \tag{16}$$

We assume $B_j = \mathrm{i}b_j$ with A_j, b_j and C_j $(j = 1, 2, 3)$ real constants left to be determined. Then solving the equations for A_j, b_j and C_j we obtain the following types of solitary wave solution.

(i) $A_j = b_j = 0\,(j = 1, 2, 3)$, $\mathrm{sgn}(\alpha_1) = \mathrm{sgn}(\alpha_2) = \mathrm{sgn}(\alpha_3)$.

In this case all three wave components are in the normal or in the anomalous regime. We obtain $C_1 = 3s_1\sqrt{\alpha_2\alpha_3}\Omega^2$, $C_2 = 3s_2\sqrt{\alpha_3\alpha_1}\Omega^2$ and $C_3 = -3s_1s_2\sqrt{\alpha_1\alpha_2}\mathrm{sgn}(\alpha_3)\Omega^2$. Thus we have the solution for equations (4)–(6):

$$a_1 = 3s_1\sqrt{\alpha_2\alpha_3}\Omega^2\mathrm{sech}^2(\Omega s - K\xi)\exp[\mathrm{i}(K_1\xi - \Omega_1 s)] \tag{17}$$

$$a_2 = 3s_2\sqrt{\alpha_3\alpha_1}\Omega^2\mathrm{sech}^2(\Omega s - K\xi)\exp[\mathrm{i}(K_2\xi - \Omega_2 s)] \tag{18}$$

$$a_3 = -3s_1s_2\sqrt{\alpha_1\alpha_2}\mathrm{sgn}(\alpha_3)\Omega^2\mathrm{sech}^2(\Omega s - K\xi)\exp[\mathrm{i}(K_3\xi - \Omega_3 s)] \tag{19}$$

where $s_j = \pm 1\,(j = 1, 2)$ and

$$K = \alpha_1\Omega\Omega_1 \tag{20}$$

$$K_1 = \frac{\alpha_1}{2}\Omega_1^2 - 2\alpha_1\Omega^2 \tag{21}$$

$$K_2 = \frac{\alpha_2}{2}\Omega_2^2 - \gamma_2\Omega_2 - 2\alpha_2\Omega^2 \tag{22}$$

$$\Omega^2 = \frac{\alpha_3\Omega_3^2 - \alpha_1\Omega_1^2 - \alpha_2\Omega_2^2 + 2(\gamma_2\Omega_2 - \gamma_3\Omega_3 - \beta)}{4(\alpha_3 - \alpha_1 - \alpha_2)}. \tag{23}$$

The parameters Ω_j $(j = 1, 2, 3)$ are given by (14)–(16). Since all solutions of the form (12) must satisfy the conditions (10), (11) and (14)–(16), we do not repeat them below. From (17)–(19) we see that all three wave components are simultaneously one-hump *bright* solitary waves with the same central position and the same travelling velocity. The physical reason for the formation of such three-wave solitary waves is due to the mutual self-trapping through a cascading process. Note that we have in fact a family of solitary wave solutions since the amplitude of each wave component may have different signs. As a particular case, when $\gamma_j = 0\,(j = 1, 2, 3)$ one has $\Omega_j = K = 0$, the solution (17)–(19) recovers that found by Werner and Drummond [7].

(ii) $b_j = 0\,(j = 1, 2, 3)$, $\mathrm{sgn}(\alpha_1) = \mathrm{sgn}(\alpha_2) = \mathrm{sgn}(\alpha_3)$.

This case gives rise to $C_1 = -3A_1/2 = 3s_1\sqrt{\alpha_2\alpha_3}\Omega^2$, $C_2 = -3A_2/2 = 3s_2\sqrt{\alpha_3\alpha_1}\Omega^2$ and $C_3 = -A_3 = -/2 - 3s_1s_2\sqrt{\alpha_1\alpha_2}\mathrm{sgn}(\alpha_3)\Omega^2$. Hence the solution now reads

$$a_1 = -3s_1\sqrt{\alpha_2\alpha_3}\Omega^2[\tfrac{2}{3} - \mathrm{sech}^2(\Omega s - K\xi)]\exp[\mathrm{i}(K_1\xi - \Omega_1 s)] \tag{24}$$

$$a_2 = -3s_2\sqrt{\alpha_3\alpha_1}\Omega^2[\tfrac{2}{3} - \mathrm{sech}^2(\Omega s - K\xi)]\exp[\mathrm{i}(K_2\xi - \Omega_2 s)] \tag{25}$$

$$a_3 = 3s_1s_2\sqrt{\alpha_1\alpha_2}\mathrm{sgn}(\alpha_3)\Omega^2[\tfrac{2}{3} - \mathrm{sech}^2(\Omega s - K\xi)]\exp[\mathrm{i}(K_3\xi - \Omega_3 s)] \tag{26}$$

where

$$K = \alpha_1\Omega\Omega_1 \tag{27}$$

$$K_1 = \frac{\alpha_1}{2}\Omega_1^2 + 2\alpha_1\Omega^2 \tag{28}$$

$$K_2 = \frac{\alpha_2}{2}\Omega_2^2 - \gamma_2\Omega_2 + 2\alpha_2\Omega^2 \tag{29}$$

$$\Omega^2 = \frac{\alpha_1\Omega_1^2 + \alpha_2\Omega_2^2 - \alpha_3\Omega_3^2 + 2(-\gamma_2\Omega_2 + \gamma_3\Omega_3 + \beta)}{4(\alpha_3 - \alpha_1 - \alpha_2)}. \tag{30}$$

The solution (24)–(26) belongs to *dark* three-wave solitary waves. However, it is quite different from conventional dark solitons found in the nonlinear Schrödinger equation in the normal dispersion regime since in the present case the intensity of each wave component has two holes. In addition, the phase difference here is absent between the background waves from the right to the left.

(iii) $A_j = 0\,(j = 1, 2, 3)$, $b_3 = C_1 = C_2 = 0$, $\alpha_1\alpha_2 > 0$, $\alpha_2\alpha_3 < 0$.

In this case we obtain $b_1 = 3s_1\sqrt{-\alpha_2\alpha_3}\Omega^2$, $b_2 = 3s_2\sqrt{-\alpha_3\alpha_1}\Omega^2$ and $C_3 = -3s_1s_2\sqrt{\alpha_1\alpha_2}\mathrm{sgn}(\alpha_3)\Omega^2$. Thus we have the solution for the equations (4)–(6):

$$a_1 = 3\mathrm{i}s_1\sqrt{-\alpha_2\alpha_3}\Omega^2\mathrm{sech}(\Omega s - K\xi)\tanh(\Omega s - K\xi)\exp[\mathrm{i}(K_1\xi - \Omega_1 s)] \quad (31)$$

$$a_2 = 3\mathrm{i}s_2\sqrt{-\alpha_3\alpha_1}\Omega^2\mathrm{sech}(\Omega s - K\xi)\tanh(\Omega s - K\xi)\exp[\mathrm{i}(K_2\xi - \Omega_2 s)] \quad (32)$$

$$a_3 = -3s_1s_2\sqrt{\alpha_1\alpha_2}\mathrm{sgn}(\alpha_3)\Omega^2\mathrm{sech}^2(\Omega s - K\xi)\exp[\mathrm{i}(K_3\xi - \Omega_3 s)] \quad (33)$$

where

$$K = \alpha_1\Omega\Omega_1 \quad (34)$$

$$K_1 = \frac{\alpha_1}{2}(\Omega_1^2 - \Omega^2) \quad (35)$$

$$K_2 = \frac{\alpha_2}{2}(\Omega_2^2 - \Omega^2) - \gamma_2\Omega_2 \quad (36)$$

$$\Omega^2 = \frac{\alpha_1\Omega_1^2 + \alpha_2\Omega_2^2 - \alpha_3\Omega_3^2 + 2(-\gamma_2\Omega_2 + \gamma_3\Omega_3 + \beta)}{2\alpha_3 + \alpha_1 + \alpha_2}. \quad (37)$$

From (31)–(33) we see that the harmonic wave (denoted by a_3) is a one-hump *bright* soliatry wave but two fundamental waves (denoted by a_1 and a_2) are (also bright) two-hump waves.

(iv) $A_j = 0\,(j = 1, 2, 3)$, $b_2 = C_1 = C_3 = 0$, $\alpha_1\alpha_2 < 0$, $\alpha_2\alpha_3 < 0$.

We find in this case $b_1 = 3s_1\sqrt{-\alpha_2\alpha_3}\Omega^2$, $b_3 = 3s_3\sqrt{-\alpha_1\alpha_2}\Omega^2$ and $C_2 = -3s_1s_3\sqrt{\alpha_3\alpha_1}\mathrm{sgn}(\alpha_3)\Omega^2$. The solution takes the form

$$a_1 = 3\mathrm{i}s_1\sqrt{-\alpha_2\alpha_3}\Omega^2\mathrm{sech}(\Omega s - K\xi)\tanh(\Omega s - K\xi)\exp[\mathrm{i}(K_1\xi - \Omega_1 s)] \quad (38)$$

$$a_2 = -3s_3s_1\sqrt{\alpha_3\alpha_1}\mathrm{sgn}(\alpha_3)\Omega^2\mathrm{sech}^2(\Omega s - K\xi)\exp[\mathrm{i}(K_2\xi - \Omega_2 s)] \quad (39)$$

$$a_3 = 3\mathrm{i}s_3\sqrt{-\alpha_1\alpha_2}\Omega^2\mathrm{sech}(\Omega s - K\xi)\tanh(\Omega s - K\xi)\exp[\mathrm{i}(K_3\xi - \Omega_3 s)] \quad (40)$$

where

$$K = \alpha_1\Omega\Omega_1 \quad (41)$$

$$K_1 = \frac{\alpha_1}{2}(\Omega_1^2 - \Omega^2) \quad (42)$$

$$K_2 = \frac{\alpha_2}{2}\Omega_2^2 - \gamma_2\Omega_2 + \gamma_2\Omega^2 \quad (43)$$

$$\Omega^2 = \frac{\alpha_3\Omega_3^2 - \alpha_1\Omega_1^2 - \alpha_2\Omega_2^2 + 2(\gamma_2\Omega_2 - \gamma_3\Omega_3 - \beta)}{2\alpha_2 - \alpha_1 - \alpha_2}. \quad (44)$$

Different from the case (iii), here one of the fundamental waves (a_2) is a one-hump solitary wave, but another fundamental wave (a_1) and the harmonic wave (a_3) are two-hump waves.

(v) $A_j = 0\,(j = 1, 2, 3)$, $b_1 = C_2 = C_3 = 0$, $\alpha_1\alpha_2 < 0$, $\alpha_2\alpha_3 > 0$.

In this case we obtain $b_2 = 3s_2\sqrt{-\alpha_3\alpha_1}\Omega^2$, $b_3 = 3s_3\sqrt{-\alpha_1\alpha_2}\Omega^2$ and $C_1 = -3s_2s_3\sqrt{\alpha_2\alpha_3}\mathrm{sgn}(\alpha_3)\Omega^2$. Thus we have

$$a_1 = -3s_2s_3\sqrt{\alpha_2\alpha_3}\mathrm{sgn}(\alpha_3)\Omega^2\mathrm{sech}^2(\Omega s - K\xi)\exp[\mathrm{i}(K_1\xi - \Omega_1 s)] \quad (45)$$

$$a_1 = 3\mathrm{i}s_2\sqrt{-\alpha_3\alpha_1}\Omega^2\mathrm{sech}(\Omega s - K\xi)\tanh(\Omega s - K\xi)\exp[\mathrm{i}(K_2\xi - \Omega_2 s)] \quad (46)$$

$$a_3 = 3\mathrm{i}s_3\sqrt{-\alpha_1\alpha_2}\Omega^2\mathrm{sech}(\Omega s - K\xi)\tanh(\Omega s - K\xi)\exp[\mathrm{i}(K_3\xi - \Omega_3 s)] \quad (47)$$

where

$$K = \alpha_1\Omega\Omega_1 \quad (48)$$

$$K_1 = \frac{\alpha_1}{2}\Omega_1^2 + \alpha_1\Omega^2 \tag{49}$$

$$K_2 = \frac{\alpha_2}{2}(\Omega_2^2 - \Omega^2) - \gamma_2\Omega_2 \tag{50}$$

$$\Omega^2 = \frac{\alpha_3\Omega_3^2 - \alpha_1\Omega_1^2 - \alpha_2\Omega_2^2 + 2(\gamma_2\Omega_2 - \gamma_3\Omega_3 - \beta)}{2\alpha_1 - \alpha_2 + \alpha_3}. \tag{51}$$

Note that the validity of the three-wave solitary wave solutions given above needs all α_j ($j = 1, 2, 3$) to be nonzero. If one of α_1, α_2 and α_3 vanishes, the solution ansatz (12) should be generalized to

$$U_j = A_j + B_j \tanh\theta + C_j \text{sech}\theta + D_j \text{sech}\theta \tanh\theta + E_j \text{sech}^2\theta \tag{52}$$

for $j = 1, 2, 3$.

In summary, we have solved the three-wave interaction equations with *dispersion*. A set of explicit three-wave solitary wave solutions has been presented. The system treated above describes a parametric process including group-velocity dispersion, and hence is valid for fairly short pulses. We can also consider in a similar way the case of a wavefield propagating along a planar waveguide with *diffraction* taken into account. The envelope equations are similar to the equations (1)–(3) but with the second-order derivatives in time to be replaced by the second-order derivatives in space. Therefore, we can also obtain spatial three-wave solitary waves with the form given above.

The formation mechanism of these one- and two-hump bright and dark three-wave solitary waves is due to the cascading effect between the three wave components. In this process, the fundamental and the harmonic waves interact with themselves through repeated wave–wave interactions. For instance the energy of one fundamental wave is first up-converted to another fundamental wave and the harmonic wave and then down-converted again, resulting in a mutual self-trapping of each wave and thus the formation of three simultaneous solitary waves. Our results may have a possible application for the design of all-optical devices [8].

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Appendix

The equations determining A_j, B_j and C_j ($j = 1, 2, 3$) are given by

$$3\alpha_1\Omega^2 C_1 - B_2^* B_3 + C_2^* C_3 = 0 \tag{53}$$

$$3\alpha_1\Omega^2 B_1 + B_2^* C_3 + C_2^* B_3 = 0 \tag{54}$$

$$-2\alpha_1\Omega^2 C_1 + A_2^* C_3 + B_2^* B_3 + C_2^* A_3 + \left(-K_1 + \frac{\alpha_1}{2}\Omega_1^2\right) C_1 = 0 \tag{55}$$

$$-\frac{\alpha_1}{2}\Omega^2 B_1 + A_2^* B_3 + B_2^* A_3 + \left(-K_1 + \frac{\alpha_1}{2}\Omega_1^2\right) B_1 = 0 \tag{56}$$

$$A_2^* A_3 + \left(-K_1 + \frac{\alpha_1}{2}\Omega_1^2\right) A_1 = 0 \tag{57}$$

$$3\alpha_2\Omega^2 C_2 - B_1^* B_3 + C_1^* C_3 = 0 \tag{58}$$

$$3\alpha_2\Omega^2 B_2 + B_1^* C_3 + C_1^* B_3 = 0 \tag{59}$$

$$-2\alpha_2\Omega^2 C_2 + A_1^* C_3 + B_1^* B_3 + C_1^* A_3 + \left(-K_2 + \frac{\alpha_2}{2}\Omega_2^2 - \gamma_2\Omega_2\right) C_2 = 0 \tag{60}$$

$$-\frac{\alpha_2}{2}\Omega^2 B_2 + A_1^* B_3 + B_1^* A_3 + \left(-K_2 + \frac{\alpha_2}{2}\Omega_2^2 - \gamma_2\Omega_2\right) B_2 = 0 \tag{61}$$

$$A_1^* A_3 + \left(-K_2 + \frac{\alpha_2}{2}\Omega_2^2 - \gamma_2\Omega_2\right) A_2 = 0 \tag{62}$$

$$3\alpha_3\Omega^2 C_3 - B_1 B_2 + C_1 C_2 = 0 \tag{63}$$

$$3\alpha_3\Omega^2 B_3 + B_1 C_2 + C_1 B_2 = 0 \tag{64}$$

$$-2\alpha_3\Omega^2 C_3 + A_1 C_2 + B_1 B_2 + C_1 A_2 + \left(-K_3 + \frac{\alpha_3}{2}\Omega_3^2 - \gamma_3\Omega_3\right) C_3 = 0 \tag{65}$$

$$-\frac{\alpha_3}{2}\Omega^2 B_3 + A_1 B_2 + B_1 A_2 + \left(-K_3 + \frac{\alpha_3}{2}\Omega_3^2 - \gamma_3\Omega_3\right) B_3 = 0 \tag{66}$$

$$A_1 A_2 + \left(-K_3 + \frac{\alpha_3}{2}\Omega_3^2 - \gamma_3\Omega_3\right) A_3 = 0. \tag{67}$$

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